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LETTER TO THE EDITOR

Coherent states and the kq-representation

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Abstract. Commuting translations in the phase plane are shown to be observables that are canonically conjugate to the annihilation operator of the harmonic oscillator states. The eigenfunctions and the eigenvalues of these commuting operators form the kq-representation. The eigenfunctions of the annihilation operator are the coherent states. The kq-eigenfunctions and the coherent states are the limiting cases of quantum-mechanical and classical wavefunctions, respectively, and it is satisfying to discover that they relate to canonically conjugate operators.

In elementary quantum mechanics the coordinate-x or the momentum-p representations are of central importance. The two operators $\hat{x}$ and $\hat{p}$ satisfy the fundamental commutation relation

$$[\hat{x}, \hat{p}] = \mathrm{i}\hbar \tag{1}$$

where $\hbar$ is the Planck constant h divided by 2π. The operators $\hat{x}$ and $\hat{p}$ and their classical counterparts are called the canonical coordinate and momentum and they are conjugate to one another in the following sense. They satisfy the commutation relation in equation (1) and each of them ($\hat{x}$ or $\hat{p}$) forms a complete set of commuting operators [1]. In this elementary case, $\hat{x}$ by itself (or $\hat{p}$ by itself) forms a complete set according to the definition of Dirac [1] because any operator that commutes with $\hat{x}$ is necessarily a function of it. Correspondingly the single operator $\hat{x}$ (or $\hat{p}$) is a complete set of commuting operators. This also means that the eigenfunctions of $\hat{x}$ (or $\hat{p}$) form a complete set of functions in the x-variable. A similar situation exists for the z-component $\hat{\ell}_z$ of the angular momentum and the angle $\hat{\varphi}$ in the xy-plane

$$[\hat{\ell}_z, \exp(\mathrm{i}\hat{\varphi})] = \hbar \exp(\mathrm{i}\hat{\varphi}) \tag{2}$$

where φ appears in the function $\exp(\mathrm{i}\varphi)$ which is periodic in φ with period 2π. For the angular coordinate, $\hat{\ell}_z$ and $\exp(\mathrm{i}\hat{\varphi})$ are the conjugate operators [2, 3]. Again each of them forms a complete set of commuting operators. However, unlike $\hat{x}$ and $\hat{p}$ whose spectrum is the whole real axis, the spectrum of $\hat{\ell}_z$ is all the integers, while the spectrum of $\exp(\mathrm{i}\hat{\varphi})$ is the unit circle. For one degree of freedom there is another pair of operators, the number and the phase, which are nearly conjugate, in the sense of equation (2), and which have attracted much attention through the years [2–4]. The word 'nearly' is used here because no unitary operator of the kind $\exp(\mathrm{i}\hat{\varphi})$ can be defined for the phase.

The notion of conjugation of $\hat{x}$ and $\hat{p}$ can also be expressed in a different way. For this we consider their exponentiated forms:

$$\exp\left(\frac{\mathrm{i}}{\hbar}\hat{x}\gamma\right) \qquad \exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}\delta\right). \tag{3}$$

It is easy to check by using the fundamental commutation relations in equation (1) that

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{x}\gamma\right)\hat{p}\exp\left(\frac{\mathrm{i}}{\hbar}\hat{x}\gamma\right) = \hat{p}+\gamma$$
$$\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{p}\delta\right)\hat{x}\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}\delta\right) = \hat{x}-\delta. \tag{4}$$

What these relations show is that $\exp((\mathrm{i}/\hbar)\hat{x}\gamma)$ is shifting $\hat{p}$ by γ, and similarly $\exp((\mathrm{i}/\hbar)\hat{p}\delta)$ shifts $\hat{x}$ and $-\delta$. One can, therefore, also say that $\hat{x}$ and $\hat{p}$ are conjugate operators if equations (4) are satisfied. Similar relations can be written for the angular momentum and angle operators in equation (2).

One often builds the annihilation operator for a Harmonic oscillator out of $\hat{x}$ and $\hat{p}$,

$$\hat{a} = \frac{1}{\lambda\sqrt{2}}\left(\hat{x}+\frac{\mathrm{i}}{\hbar}\lambda^2\hat{p}\right) \tag{5}$$

with λ being a constant. This operator is closely related to coherent states which were discovered in the very beginning of quantum mechanics [5] and which acquired much popularity with their application to quantum optics [6]. By definition, coherent states $|\alpha\rangle$ are eigenstates of the annihilation operator $\hat{a}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{6}$$

where α is a complex number given by

$$\alpha = \frac{1}{\lambda\sqrt{2}}\left(\bar{x}+\frac{\mathrm{i}}{\hbar}\lambda^2\bar{p}\right) \tag{7}$$

with $\bar{x}$ and $\bar{p}$ being the coordinate and momentum expectation values in the coherent state $|\alpha\rangle$. Coherent states are of much importance in quantum physics and there have been numerous books written about them [7].

In view of the importance of coherent states and in view of the fact that they are eigenstates of the annihilation operator $\hat{a}$ (equation (6)) we look in this letter for a complete set of commuting operators that is conjugate to $\hat{a}$. As is well known, together with $\hat{a}$ one usually defines its Hermitian conjugate $\hat{a}^\dagger$ and they satisfy the commutation relation [6]

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{8}$$

This commutator looks similar to the relation between $\hat{x}$ and $\hat{p}$ in equation (1). However, $\hat{a}^\dagger$ is not the conjugate operator to $\hat{a}$ because $\hat{a}^\dagger$ has no eigenstates [8]. In this letter we prove that the complete set of commuting translations in the phase plane is conjugate to $\hat{a}$. This is the same set of translations which defines the kq-representation [9], and our proof establishes, therefore, a close relationship between the coherent states and the kq-representation.

The proof is entirely elementary and it starts with the shift operator $D(\alpha)$:

$$D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = \exp\left[\frac{\mathrm{i}}{\hbar}(\bar{p}\hat{x}-\bar{x}\hat{p})\right] = \hat{D}^\dagger(-\alpha) \tag{9}$$

where α^* is the complex conjugate of α in equation (7). The annihilation operator $\hat{a}$ and the shift $\hat{D}(\alpha)$ satisfy the following commutation relation [6, 8]

$$[\hat{a}, \hat{D}(\alpha)] = \alpha\hat{D}(\alpha). \tag{10}$$

This relation looks similar to equation (2) for the angular coordinate. However, $\hat{D}(\alpha)$ for a given α is not a complete set of commuting operators. This is seen from the commutation relation for two shift operators $\hat{D}(\alpha)$ and $\hat{D}(\beta)$:

$$\hat{D}(\alpha)\hat{D}(\beta) = \exp(\alpha\beta^* - \alpha^*\beta)\hat{D}(\beta)\hat{D}(\alpha). \tag{11}$$

When $\alpha\beta^* - \alpha^*\beta = 2\pi \mathrm{i} n$ with integer n, the operator $\hat{D}(\beta)$ commutes with $\hat{D}(\alpha)$. This means that $\hat{D}(\alpha)$ by itself is not a complete set of commuting operators. One can, however, check that the set of operators $\hat{D}(\alpha_{mn})$

$$\hat{D}(\alpha_{mn}) = (-1)^{mn} \exp\left(-\frac{\mathrm{i}}{\hbar}\hat{p}mb\right) \exp\left(\mathrm{i}\hat{x}n\frac{2\pi}{b}\right) \tag{12}$$

for an arbitrary constant b is a complete commuting set of operators. They are obtained from equation (9) by substituting α_{mn} for α

$$\alpha_{mn} = \frac{1}{\lambda\sqrt{2}}\left(mb + \mathrm{i}\lambda^2 n\frac{2\pi}{b}\right). \tag{13}$$

This fact is known for the kq-representation [9], for which the kq-coordinates determine the eigenvalues of the complete set of the commuting operators in equation (12). We have

$$\hat{D}(\alpha_{mn})|kq\rangle = (-1)^{mn} \exp\left(-\mathrm{i}kmb + \mathrm{i}qn\frac{2\pi}{b}\right)|kq\rangle \tag{14}$$

Here k is the quasi-momentum and it varies from 0 to $2\pi/b$ and q is the quasi-coordinate which varies between 0 and b. Equation (10) can also be rewritten as follows with α replaced by α_{mn}:

$$[\hat{a}, \hat{D}(\alpha_{mn})] = \alpha_{mn}\hat{D}(\alpha_{mn}). \tag{15}$$

Since $\hat{D}(\alpha_{mn})$ is a complete set of commuting operators, for α_{mn} given by equation (13), we can now claim that $\hat{D}(\alpha_{mn})$ is the conjugate set to the annihilation operator $\hat{a}$: equation (15) for $\hat{a}$ and $\hat{D}(\alpha_{mn})$ is of the same kind as equations (1) and (2) for the elementary operators in quantum mechanics, and each of the operators $\hat{a}$ and $\hat{D}(\alpha_{mn})$ form a complete set of commuting operators. As is well known (see later) $\hat{a}$ is overcomplete. We can rewrite equation (15) so that it assumes the form of equations (4). We have

$$\hat{D}^{\dagger}(\alpha_{mn})\hat{a}\hat{D}(\alpha_{mn}) = \hat{a} + \alpha_{mn} \tag{16}$$

where $\hat{D}^{\dagger}(\alpha)$ is given in equation (9).

This shows that $\hat{D}(\alpha_{mn})$ is a shift operator of $\hat{a}$ in exactly the same way as was shown for $\hat{x}$ and $\hat{p}$ (equations (4)). With equation (16) at hand we can state that $\hat{D}(\alpha_{mn})$ is a complete commuting set of operators which is conjugate to the annihilation operator $\hat{a}$. The conjugacy of $\hat{D}(\alpha_{mn})$ to $\hat{a}$ can also be seen on a qualitative level. The eigenstates of $\hat{a}$ are the coherent states in which x and p are optimally localized [5, 6, 8]. On the other hand, in the eigenstates $|kq\rangle$ of $\hat{D}(\alpha_{mn})$ (equation (14)) both x and p are completely delocalized. This feature is similar to the one for the operators $\hat{x}$ and $\hat{p}$ in equation (1): for states in which x is well localized, p is delocalized, and *vice versa*, when p is very well localized, x is poorly determined. The similarity between the operators in equations (1) and (15) persists also in other ways but, in general, there are also differences which stem from the non-Hermiticity of the annihilation operator $\hat{a}$. Let us consider these differences in more detail. The transformation $\langle x|p\rangle$ between the x and p-representations is [1]

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{\mathrm{i}}{\hbar}xp\right). \tag{17}$$

The square of the absolute value $|\langle x|p\rangle|^2$ is a constant equal to $1/2\pi\hbar$ and is proportional to the probability of measuring x in the $|x\rangle$-state or of measuring p in the $|p\rangle$-state. This shows that for the operators $\hat{x}$ and $\hat{p}$, which are both Hermitian, when one of them is well defined the other one is completely undetermined. For the conjugate operators $\hat{a}$ and $\hat{D}(\alpha_{mn})$ the situation is different because the annihilation operator $\hat{a}$ is not Hermitian and its eigenstates (equation (6)) form a highly overcomplete system of states [6, 8]. As was first suggested by von Neumann [10] and later proven in a number of publications [11–13] the discrete set of coherent states $|\alpha_{mn}\rangle$ with α_{mn} as given in equation (13) is complete. The $|\alpha_{mn}\rangle$-states are centred on a rectangular lattice in the xp-plane with a unit cell area of h, the Planck constant. This is the von Neumann lattice [14]. Having in mind that $D(\alpha)|0\rangle = |\alpha\rangle$, one can see from equation (14) that the set $|\alpha_{mn}\rangle$ assumes a simple form when written in the kq-representation. (See the expression for general α, $\langle kq|\alpha\rangle$) [13]

$$\langle kq|\alpha_{mn}\rangle = (-1)^{mn} \exp\left(-\mathrm{i}kmb + \mathrm{i}qn\frac{2\pi}{b}\right)\langle kq|0\rangle \tag{18}$$

where $\langle kq|0\rangle$ is the ground state of the harmonic oscillator in the kq-representation. The latter is [9]

$$\begin{aligned}\langle kq|0\rangle &= \left(\frac{a}{2\pi}\right)^{1/2}\sum_n \exp(\mathrm{i}kbn)\psi_0(q-nb)\\ &= \left(\frac{b}{2\lambda\pi^{3/2}}\right)^{1/2}\sum_n \exp(\mathrm{i}kbn)\exp\left[-\frac{(q-nb)^2}{2\lambda^2}\right]\end{aligned} \tag{19}$$

with

$$\psi_0(x) = \left(\frac{1}{\lambda\pi^{1/2}}\right)^{1/2}\exp\left(-\frac{x^2}{2\lambda^2}\right)$$

(λ a constant) being the ground state of the harmonic oscillator in the x-representation. The exponential term in equation (18) is similar to the function in equation (17): the kq-coordinates and the integers m and n appear in the exponent of equation (18) in the same way as x and p appear in (17). From this point of view there is a full analogy between the conjugate pairs $\hat{x}$, $\hat{p}$ and $\hat{a}$, $\hat{D}(\alpha_{mn})$. However, despite these similarities, the eigenstates $|\alpha_{mn}\rangle$ for different m and n are not orthogonal, and the square of the absolute value of the function $\langle kq|\alpha_{mn}\rangle$ in (18) is not a constant as in the case of $\langle x|p\rangle$ (17). From equation (18) we see that $|\langle kq|\alpha_{mn}\rangle|^2$ is

$$|\langle kq|\alpha_{mn}\rangle|^2 = |\langle kq|0\rangle|^2. \tag{20}$$

It does not depend on m and n but it depends on k and q. In this connection the following remark can be made. For the operators $\hat{x}$ and $\hat{p}$ (both Hermitian), the shift relations hold for both of them as given by equation (4). On the other hand, for the conjugate operators $\hat{a}$ and $\hat{D}(\alpha_{mn})$ we have written only one shift relation with $\hat{D}(\alpha_{mn})$ being the shift operator of $\hat{a}$ (equation (16)). The reason for this is that $\hat{D}(\alpha_{mn})$ is a unitary operator and is similar to the operators in equation (3). However, since $\hat{a}$ is not Hermitian, its exponentiation does not lead to a unitary shift operator. We see, therefore, that while full symmetry exists for the x and p conjugate operators, there is no such symmetry for the conjugate operators for $\hat{a}$ and $\hat{D}(\alpha_{mn})$.

Having established that the set $\hat{D}(\alpha_{mn})$ is conjugate to the well known annihilation operator $\hat{a}$, it is of interest to consider more closely the properties and the significance of $\hat{D}(\alpha_{mn})$. It is simple to check that for any two states $|1\rangle$ and $|2\rangle$ the wavefunctions $\psi_1(x)$ and $\psi_2(x)$ in the x-representation can be written

$$\psi_1^*(x)\psi_2(x) = \frac{1}{2\pi\hbar}\int \exp\left(\frac{\mathrm{i}}{\hbar}px\right)\langle 1|\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{x}p\right)|2\rangle\,\mathrm{d}p. \tag{21}$$

Similarly, for their Fourier transforms $F_1(p)$ and $F_2(p)$ we have

$$F_1^*(p)F_2(p) = \frac{1}{2\pi\hbar}\int \exp\left(-\frac{\mathrm{i}}{\hbar}px\right)\langle 1|\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}x\right)|2\rangle\,\mathrm{d}p. \tag{22}$$

From equation (21) we see that for the product of the wavefunctions $\psi_1^*(x)\psi_2(x)$ in the x-representation, the shift operator $\exp(-(\mathrm{i}/\hbar)\hat{x}p)$ in p-space appears in the sandwich between the states $|1\rangle$ and $|2\rangle$ on the right-hand side of this equation. Similarly, for the product $F_1^*(p)F_2(p)$ in equation (22) the shift operator $\exp((\mathrm{i}/\hbar)\hat{p}x)$ in x-space appears in the sandwich on the right-hand side. One should expect that, for the product $C_1^*(k,q)C_2(k,q)$ of two wavefunctions in the kq-representation, the shift operator $\hat{D}(\alpha_{mn})$ in equation (12) will appear, because it leads to discrete shifts both in x and p-directions. Indeed, one can show that [14]

$$\begin{aligned} C_1^*(k,q)C_2(k,q) &= \frac{1}{2\pi}\sum_{m,n}(-1)^{mn}\langle 1|\hat{D}^\dagger(\alpha_{mn})|2\rangle \exp\left(\mathrm{i}q\frac{2\pi}{b}n - \mathrm{i}kbm\right) \\ &= \frac{1}{2\pi}\sum_{m,n}\langle 1|\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{p}mb\right)\exp\left(\mathrm{i}\hat{x}n\frac{2\pi}{b}\right)|2\rangle \exp\left(\mathrm{i}q\frac{2\pi}{b}n - \mathrm{i}kbm\right). \end{aligned} \quad (23)$$

The expressions in equations (21)–(23) give interference terms between the states $|1\rangle$ and $|2\rangle$ in the x, p and kq-representations, respectively.

In the context of this paper, of particular interest is equation (23) which we rewrite for the case $|1\rangle = |2\rangle = |\alpha\rangle$ ($|\alpha\rangle$ being the coherent state):

$$|\langle kq|\alpha\rangle|^2 \equiv |C_\alpha(k,q)|^2 = \frac{1}{2\pi}\sum_{m,n}(-1)^{mn}\langle\alpha|\hat{D}^\dagger(\alpha_{mn})|\alpha\rangle \exp\left(\mathrm{i}q\frac{2\pi}{b}n - \mathrm{i}kbm\right). \quad (24)$$

By using the following formulae [6]

$$\hat{D}^\dagger(\alpha_{mn})\hat{D}(\alpha) = D(\alpha - \alpha_{mn})\exp[-\tfrac{1}{2}(\alpha^*\alpha_{mn} - \alpha\alpha_{mn}^*)] \quad (25)$$

$$\langle\alpha|\beta\rangle = \exp(\alpha^*\beta - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2) \quad (26)$$

equation (24) becomes (with $b^2/2\pi\lambda^2 = 1$)

$$|\langle kq|\alpha\rangle|^2 = \frac{1}{2\pi}\sum_{m,n}(-1)^{mn}\exp\left[-\frac{\pi}{2}(m^2+n^2)\right]\exp\left[\mathrm{i}\frac{2\pi}{b}(q-\bar{x})n - \mathrm{i}b\left(k - \frac{\bar{p}}{\hbar}\right)m\right]. \quad (27)$$

Here $\bar{x}$ and $\bar{p}$ appear in the definition of α in equation (7), and have the meaning of the average coordinate and of the average momentum, respectively, in the coherent state $|\alpha\rangle$. In view of the fact that $\hat{a}$ and $\hat{D}(\alpha_{mn})$ are conjugate operators (equations (15) and (16)) it is of interest to consider the result in equation (27) in more detail. As was pointed out before, the von Neumann set of coherent states $|\alpha_{st}\rangle$ with s and t integers, is complete (see equation (18)) [10–13]. It is easy to check that the expression in equation (27) does not change when $\bar{x}$ goes into $\bar{x} + sb$, and $\bar{p}$ goes into $\bar{p} + t\hbar(2\pi/b)$:

$$|\langle kq|\bar{x} + sb, \bar{p} + t\hbar(2\pi/b)\rangle|^2 = |\langle kq|\bar{x}, \bar{p}\rangle|^2. \quad (28)$$

This is a generalization of the result in equation (20) and is a consequence of the periodicity of $|\langle kq|\bar{x},\bar{p}\rangle|^2$ in $\bar{x}$ and $\bar{p}$. One can also rewrite equation (27) in the following way,

$$|\langle kq|\bar{x},\bar{p}\rangle|^2 = |\langle k - (\bar{p}/\hbar), q - \bar{x}|0\rangle|^2 \quad (29)$$

where by $|0\rangle$ we denote the ground state of the harmonic oscillator (equations (18) and (19)). As is well known [13, 15], the wavefunction $\langle k,q|0\rangle$ has a zero at $k = \pi/b$, $q = b/2$. In [15] a plot is given of $|\langle kq|0\rangle|$ which shows that $|\langle kq|0\rangle|^2$ has a maximum at $k = q = 0$. The result in equations (27) and (29) demonstrates, therefore, a very interesting feature of the coherent state $|\alpha\rangle \equiv |\bar{x},\bar{p}\rangle$ in the kq-representation: the probability of measuring the quasi-momentum k and the quasi-coordinate q is maximum when $k = \bar{p}/\hbar$, $q = \bar{x}$ and it vanishes when $k-(\bar{p}/\hbar) = \pi/b, q-\bar{x} = b/2$. This leads us to a new physical (non-intuitive) interpretation of the zero of the kq-function $\langle kq|0\rangle$. According to equation (29), k and q in $\langle kq|0\rangle$ can be considered as the differences between the quasi and the average coordinates and momenta. The zero of $\langle kq|0\rangle$ then means that the coherent state $|\bar{x}, \bar{p}\rangle$ has no kq-component for $q-\bar{x} = b/2, k-(\bar{p}/\hbar) = \pi/b$. For another physical interpretation of a zero of the kq-function the reader is referred to [16] in connection with the quantum Hall effect. As was mentioned above, the variation limits of k and q are $0 \leqslant k \leqslant 2\pi/b, 0 \leqslant q \leqslant b$. This defines the von Neumann–Gabor unit cell in the phase plane [17]. As is seen from equation (28) the probability of measuring k and q in any coherent state is fully determined by its values in a von Neumann-Gabor unit cell.

In conclusion, it is shown that the annihilation operator $\hat{a}$ for coherent states and the commuting translations $\hat{D}(\alpha_{mn})$ in the phase plane, which define the kq-representation, form a set of conjugate operators. The eigenstates $|\alpha\rangle \equiv |\bar{x}, \bar{p}\rangle$ of $\hat{a}$ are the most localized states in coordinate x and momentum p, while the eigenstates $|kq\rangle$ of $\hat{D}(\alpha_{mn})$ exhibit no localization whatsoever in x and p. The product of uncertainties assumes the minimum in a coherent state, $\Delta x \Delta p = \hbar/2$. In a kq-state this product is infinite. It is very satisfying to discover that $\hat{a}$ and $\hat{D}(\alpha_{mn})$ are conjugate operators.

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